Population point process model of antibunched light and its spectrum



1989 J. Phys. A: Math. Gen. 22 L259

(http://iopscience.iop.org/0305-4470/22/7/002)

J. Phys. A: Math. Gen. 22 (1989) L259–L263. Printed in the UK

LETTER TO THE EDITOR

# Population point process model of antibunched light and its spectrum

S K Srinivasan

Department of Mathematics, Indian Institute of Technology, Madras 600 036, India

Received 25 November 1988

**Abstract.** The letter proposes a model of population of photons in which emissions occur through a process of non-Markov evolution. The photon population statistics are shown to be antibunched in their character with intensity correlation characteristic of resonant fluorescence and other associated phenomena.

The object of this letter is to demonstrate the possibility of accommodating antibunched light within the framework of the population point process model of cavity photons [1–4]. Although antibunching was confirmed around 1977 by the measurement of photon statistics in resonance fluorescence [5], there are still many attempts to measure the statistics of photocounts of other sources of light such as depleted cascade emission [6] or radiation from atoms excited by a space-charge-limited electron stream [7, 8]. There are now available several surveys [9–11] dealing exclusively with antibunching and the topic continues to be interesting in the broad area of non-linear optics particularly in view of its connection to other phenomena like sub-Poissonian behaviour and squeezing [12]. Since population models form the basis for the description of various characteristics like thermal behaviour and lasing, it is considered worthwhile to report some recent findings that would enable us to conclude that such models are viable enough to yield information on spectral properties of the resulting radiation.

The starting point is the population model of Shimoda *et al* [1] as modified by Shepherd [2] and the author [3, 4] to include the detection process. To take into account the non-Markov effect arising from non-uniformity in the time rate of emission, we modify the processes of stimulated and spontaneous emission. To be precise, we assume that the process of spontaneous emission taken in isolation forms an ordinary renewal process whose interval spans are sums of $m$ positive independent random variables each with a negative exponential distribution. To be specific we choose $m=3$ and the parameters as $\nu_1$, $\nu_2$ and $\nu_3$. Likewise the stimulated emissions due to any single photon can be thought of as a general point process of emissions in which the time to the first emission is the sum of $(n+1)$ positive independent random variables each with a negative exponential distribution with parameter $\lambda_i$, with subsequent emissions occurring at a constant rate $\alpha=\lambda_{n+1}$. Thus each of the photons is assumed to evolve in time, independent of each other, through a series of phases, the sojourn through the first $n$ phases being completed before the actual emission takes place, the rate of emission itself being a constant $\alpha$ in the final (residuary) phase. The emitted photons, in turn repeat the process, independent of other photons. The (cavity) absorption rate (which is identified as the death rate per individual photon) is assumed to be a constant equal to $\mu$ in all the phases except the last where it is taken to be equal to $\lambda_{n+1}+\mu$. This particular differential choice of the parameter is to facilitate thermal equilibrium wherever it is needed. The field detector interaction is modelled as an emigration process with a constant rate $\eta$ per individual irrespective of its state of evolution in phases. In the original model of Shimoda *et al* [1], the parameters $m$ and $n$ correspond to the values 1 and 0 respectively and $\nu$, the parameter of the exponential distribution, is indeed the parameter of the Poisson process which characterises the spontaneous emission process. It is to be noted that the evolution through phases is only a device to handle the non-Markov emission process. We now proceed to analyse the model with special reference to the detection process so that the intensity correlation can be explicitly characterised. For convenience we choose $n=2$ and introduce the following notation:

$Z(t)$: the state process of spontaneous emission (immigration) taking values 1, 2, 3 corresponding to $m=3$;

$X_i(t)$: the size of the cavity photon population in phase $i$ $(i=1,2)$;

$X(t)$: the total size of the cavity photon population;

$$g_i(w,t)=E[w^{X(t)}|X(0)=X_i(0)=1,\ \nu_1=\nu_2=\nu_3=0] \qquad (1)$$

$$G_i(w,t)=E[w^{X(t)}|Z(0)=i,\ X(0)=0] \qquad (2)$$

where $E$ stands for the mathematical expectation of the quantity within the brackets. The generating function defined by (1), specially conditioned, corresponds to the immigration taboo process and facilitates the solution. The exponential nature of the lifespan of the phases leads to the following differential equations [3, 4]:

$$\frac{\partial g_i(w,t)}{\partial t}=-(\lambda_i+\mu+\eta)g_1(w,t)+\lambda_i g_2(w,t)+\mu+\eta \qquad i=1,2, \qquad (3)$$

$$\frac{\partial g_3(w,t)}{\partial t}=-(\lambda_3+\mu+\eta+\alpha)g_3(w,t)+\alpha g_3(w,t)g_1(w,t)+\mu+\eta+\lambda_3 \qquad (4)$$

$$\begin{aligned}\frac{\partial G_i(w,t)}{\partial t}&=-\nu_i G_i(w,t)+\nu_i G_{i+1}(w,t) \qquad i=1,2\\ \frac{\partial G_3(w,t)}{\partial t}&=-\nu_3 G_3(w,t)+\nu_3 G_1(w,t)g_1(w,t)\end{aligned} \qquad (5)$$

with the initial conditions

$$g_i(w,0)=w \qquad G_i(w,0)=1 \qquad i=1,2,3. \qquad (6)$$

For purposes of illustration we further set

$$\lambda_1=\lambda_2=\lambda_3=\lambda=\alpha \qquad \nu_1=\nu_2=\nu_3=\nu.$$

In order to characterise the detection process, we need the equilibrium distribution of cavity photons as well as the first few moments of $g_i(w,t)$. The detection process is best characterised by the two time correlation $h_2(t_1, t_2)$ where $h_2(t_1, t_2)\mathrm{d}t_1\mathrm{d}t_2$ represents the inclusive probability that a photon is detected in each of the intervals $(t_1, t_1+\mathrm{d}t_1)$ and $(t_2, t_2, t_2+\mathrm{d}t_2)$ and an experimentally significant measure is the equilibrium value $h_{\mathrm{sty}}(t)$ of $h_2(t_1, t_2)$ given by

$$\begin{aligned}h_{\mathrm{sty}}(t)&=\lim_{t_1,t_2\to\infty} h_2(t_1,t_2)\\ |t_2-t_1|&=t.\end{aligned} \qquad (7)$$

In addition we also need the equilibrium first-order function $h_1(\cdot)$ where

$$h_1(\cdot) = \lim_{t\to\infty} h_1(t) \tag{8}$$

represents the stationary rate of detection and is given by

$$h_1(\cdot) = \lim_{t\to\infty} \eta E[X(t)]. \tag{9}$$

Introducing the notation

$$\begin{aligned} A_i(t) &= \left.\frac{\partial G_i}{\partial w}\right|_{w=1} & B_i(t) &= \left.\frac{\partial^2 G_i}{\partial w}\right|_{w=1} \\ a_i(t) &= \left.\frac{\partial g_i}{\partial w}\right|_{w=1} & b_i(t) &= \left.\frac{\partial^2 g_i}{\partial w^2}\right|_{w=1} \end{aligned} \tag{10}$$

we find, by the use of combinatorial arguments,

$$h_1(\cdot) = \eta A_1(\infty) \tag{11}$$

$$h_{\rm sty}(t) = h_1(\cdot) h_1^{\rm CE}(t) + B_1(\infty)\eta^2 a_1(t) \tag{12}$$

where $h_1^{\rm CE}(t)$ is the inclusive probability of a detection in an infinitesimal interval of time following $t$, conditional on the cavity population being in equilibrium at the (time) origin where a photon has been detected by the detector. Thus our objective would be accomplished if we obtain the moments $A_i(t)$ and $B_i(t)$ and the function $h_1^{\rm CE}(t)$.

The moments can be obtained by a straightforward process of differentiation of equations (3)-(5) and solving them by the use of the Laplace transform technique. We skip the details and give the final Laplace transform solution:

$$\begin{aligned} a_i(t) &= e^{-(\mu+\eta)t} \\ A_i^*(s) &= \frac{a_1^*(s)\nu^{4-i}(\nu+s)^{i-1}}{s(3\nu^2+3\nu s+s^2)} \\ B_1^*(s) &= \frac{\nu^3[2A_1^*(\mu+\eta+s)+b_1^*(s)]}{s(3\nu^2+3\nu s+s^2)} \\ b_1^*(s) &= \lambda^2[s+4\lambda+2\mu+2\eta]/[D(s)D(s/2)] \\ D(s) &= (s+\lambda+\eta+\mu)^2-\lambda^2 \end{aligned} \tag{13}$$

where we have used $*$ as a superscipt to denote the Laplace transform. An explicit expression for the function $h_1^{\rm CE}(t)$ can be obtained by the use of combinatorial arguments:

$$h_1^{\rm CE}(t) = \frac{\eta[(\nu+\eta+\mu)^2 A_1(t) + (\nu+\mu+\eta)A_2(t) + \nu^2 A_3(t)]}{(\mu+\eta)^2 - 3\nu(\mu+\eta) + 3\nu^2}. \tag{14}$$

Using (13) and (14), we finally obtain, after some calculations,

$$h_1(\cdot) = \frac{\eta}{3K} \tag{15}$$

$$h_{\rm sty}(t) = \left(\frac{\eta}{3K}\right)^2 + \left(B_1(\infty) - \frac{3+K^2}{3K^2 D_+ D_-}\right)\eta^2\, e^{-}(\mu+\eta)t_{+\eta^2}J_1(t) \tag{16}$$

where

$$B_1(\infty)=\frac{1}{3K^2}\left(\frac{2}{K^2+3K+3}+\frac{L^2}{2L+K}\right) \tag{17}$$

$$D_\pm=K^2\pm3K+3 \qquad K=(\mu+\eta)/\nu \qquad L=\lambda/\nu \tag{18}$$

$$J_1(t)=\mathrm{e}^{-3\nu t/2}[-K^3\sqrt{3}\sin\sqrt{3}t\nu/2+K(6-K^2)\cos\sqrt{3}t\nu/2]/(9LD_+D_-). \tag{19}$$

Equations (15) and (16) are the main results from which many useful conclusions can be drawn. A measure of bunching $\mathcal{B}$ can be introduced by [4, 16]

$$\mathcal{B}=h_{\mathrm{sty}}(0)/h_{\mathrm{sty}}(\infty) \tag{20}$$

and we find

$$\mathcal{B}=9K^2B_1(\infty)=3\left(\frac{2}{K^2+3K+3}+\frac{L^2}{2L+K}\right) \tag{21}$$

from which we readily conclude that there are many physically feasible choices of $K$ and $L$ for which $\mathcal{B}$ is less than unity, leading to antibunching. For such a choice, (16) shows that apart from a constant the correlation consists of a pure exponential function with argument $-K\nu t$ and an exponential function with argument $-3\nu t/2$ modulated by sine and cosine functions. Such correlations were noticed earlier in the literature in the context of resonance fluorescence and other associated phenomena [5, 13–15]. It is also interesting to note that intensity correlations with further modulations can be accommodated by increasing the value of $m$.

A special case of interest arises when $L=0$. This is a generalisation of the coherent model of Shepherd [2]. In that case we can eliminate the pure exponential term by setting $K=5.45$ in which case $h_{\mathrm{sty}}(\cdot)$ is given by

$$h_{\mathrm{sty}}(t)=\left(-\frac{\eta}{3K}\right)^2\left[1-\left(1.91\sin\frac{\sqrt{3}\nu t}{2}+0.88\cos\frac{\sqrt{3}\nu t}{2}\right)\mathrm{e}^{-3\nu t/2}\right] \tag{22}$$

and the bunching factor is nearly one eighth. Thus the population point process model is viable enough to describe many facets of the cavity radiation and characterise the photon statistics.

Finally we observe that the structure of the correlation function as described by (16) remains the same for the general case in which an arbitrary number $(n+1)$ of phases (in the process of stimulated emission) are introduced. In this case the formula for $B_1(\infty)$ as given by (17) still holds good provided we replace the second term within the brackets by $L^n/[(K+L)^n-L^n]$. Further details relating to the structure of the various formulae presented above will be published elsewhere.

## References


[1] Shimoda K, Takahasi H and Townes C H 1957 *J. Phys. Soc. Japan* **12** 686–700

[2] Shepherd T J 1981 *Opt. Acta* **28** 567–83

[3] Srinivasan S K 1986 *J. Phys. A: Math. Gen.* **19** L513–6

[4] Srinivasan S K 1988 *Point Process Models of Cavity Radiation and Detection* (London: Griffin)

[5] Kimble H J, Dagenais M and Mandel L 1977 *Phys. Rev. Lett.* **39** 691–5

[6] Saleh B E A and Teich M C 1985 *Opt. Commun.* **52** 429–32

[7] Jakeman E and Walker J G 1985 *Opt. Acta* **32** 1303–8

[8] Teich M C and Saleh B E A 1985 *J. Opt. Soc. Am.* **B2** 275–82

[9] Loudon R 1980 *Rep. Prog. Phys.* **43** 913-49
[10] Paul H 1982 *Rev. Mod. Phys.* **54** 1061-2
[11] Teich M C and Saleh B E A 1988 *Progress in Optics* vol XXVI, ed E Wolf (Amsterdam: North-Holland) pp 1-104
[12] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709-59
[13] Senitzky I R 1972 *Phys. Rev.* A **6** 1171-4
[14] Carmichael H J and Walls D F 1976 *J. Phys. B: At. Mol. Phys.* **9** L43-6
[15] Mandel L 1979 *Opt. Lett.* **4** 205-7
[16] Srinivasan S K 1986 *J. Phys. A: Math. Gen.* **19** L595-8